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A remark on Martin's maximum

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1 Introduction

In this paper we prove that MM, Martin's maximum, implies the partial square principle at ω_1. First we recall the partial square principle:

Definition 1.1. *Let κ be an uncountable cardinal. For $S \subseteq \mathrm{Lim}(\kappa^+)$ let*

$$\square_\kappa(S) \equiv \text{There exists a sequence } \langle c_\alpha \mid \alpha \in S\rangle \text{ such that}$$
$$\text{(i) } c_\alpha \text{ is a club of } \alpha \text{ with } \mathrm{o.t.}(c_\alpha) \leq \kappa \text{ for each } \alpha \in S,$$
$$\text{(ii) if } \alpha \in S \text{ and } \beta \in \mathrm{Lim}(c_\alpha), \text{ then } \beta \in S \text{ and } c_\beta = c_\alpha \cap \beta.$$

A sequence $\langle c_\alpha \mid \alpha \in S\rangle$ satisfying (i) and (ii) is called a $\square_\kappa(S)$-sequence.

In the above

$$\begin{aligned} \mathrm{Lim}(A) &:= \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}\,, \\ \mathrm{o.t.}(A) &:= \text{the order type of } A\,, \end{aligned}$$

for a set A of ordinals.

Note that $\square_\kappa(\mathrm{Lim}(\kappa^+))$ is equivalent to Jensen's $\square_\kappa$ introduced in [2]. Hence $\square_\kappa(\mathrm{Lim}(\kappa^+))$ holds for every uncountable cardinal κ in L. It is not hard to see that if S is a nonstationary subset of κ^+, then $\square_\kappa(S)$ holds. Moreover it is shown by Shelah [3] that if μ and κ are regular cardinals with $\mu < \kappa$, then there exists $S \subseteq \mathrm{Lim}(\kappa^+)$ such that the set $\{\alpha \in S \mid \mathrm{cf}(\alpha) = \mu\}$ is stationary and $\square_\kappa(S)$ holds.

On the other hand it is known that if κ is a regular uncountable cardinal and there exists a weakly compact cardinal above κ, then there exists a $<\kappa$-closed forcing extension in which $\square_\kappa(S)$ fails for every $S \subseteq \mathrm{Lim}(\kappa^+)$ such that the set $\{\alpha \in S \mid \mathrm{cf}(\alpha) = \kappa\}$ is stationary. In particular it is independent of ZFC whether there exists $S \subseteq \mathrm{Lim}(\omega_2)$ such that $\{\alpha \in S \mid \mathrm{cf}(\alpha) = \omega_1\}$ is stationary and $\square_{\omega_1}(S)$ holds. In this paper we prove that MM implies the existence of such $S \subseteq \mathrm{Lim}(\omega_2)$:

Theorem 1.5. *Assume* MM. *Then there exists $S \subseteq \mathrm{Lim}(\omega_2)$ such that the set $\{\alpha \in S \mid \mathrm{cf}(\alpha) = \omega_1\}$ is stationary and $\square_{\omega_1}(S)$ holds.*

Below we let $\square(S)$ denote $\square_{\omega_1}(S)$. Moreover we let

$$E^2_i := \{\alpha \in \omega_2 \mid \mathrm{cf}(\alpha) = \omega_i\}$$

for $i = 0, 1$.

2 Preliminaries

2.1 Facts on ω_1-stationary preserving σ-Baire poset

A poset $\mathbb{P}$ is said to be ω_1-stationary preserving if every stationary subset of ω_1 remains to be stationary in every generic extension by $\mathbb{P}$. $\mathbb{P}$ is said to be σ-Baire if the forcing extension by $\mathbb{P}$ adds no new sequences of ordinals of length ω.

In the proof of Thm.1.5 we will construct an ω_1-stationary preserving σ-Baire poset and apply MM to it. Here we present two facts on such poset.

The first one is a fact, essentially due to Woodin [4], on a consequence of MM applied to such poset:

Definition 2.1. *Let $\mathbb{P}$ be a poset and M be a set. g is called an $(M, \mathbb{P})$-generic filter if g is a filter on $\mathbb{P} \cap M$ such that $g \cap A \cap M \neq \emptyset$ for every maximal antichain $A \in M$ in $\mathbb{P}$.*

Fact 2.2. *Assume* MM. *Suppose that $\mathbb{P}$ is an ω_1-stationary preserving σ-Baire poset and that θ is a sufficiently large regular cardinal with $\theta^\omega = \theta$. Then the set of all $M \in [\mathcal{H}_\theta]^{\omega_1}$ such that*

(i) M is internally approachable of length ω_1,

(ii) there exists an $(M, \mathbb{P})$-generic filter,

is stationary in $[\mathcal{H}_\theta]^{\omega_1}$.

The second one is a sufficient condition for a poset to be ω_1-stationary preserving and σ-Baire:

Definition 2.3. *Let W be a set with $\omega_1 \subseteq W$. $X \subseteq [W]^\omega$ is said to be projectively stationary if the set $\{x \in X \mid x \cap \omega_1 \in H\}$ is stationary in $[W]^\omega$ for every stationary $H \subseteq \omega_1$.*

Definition 2.4. *Let $\mathbb{P}$ be a poset and M be a set. $p \in \mathbb{P}$ is called a strongly $(M, \mathbb{P})$-generic condition if for every maximal antichain $A \in M$ in $\mathbb{P}$ there exists $q \in A \cap M$ with $q \geq p$.*

Fact 2.5. *Let $\mathbb{P}$ be a poset. Suppose that the following holds:*

($*$) For every sufficiently large regular cardinal θ and every $q \in \mathbb{P}$ the set

$$\{M \in [\mathcal{H}_\theta]^\omega \mid \text{a strongly } (M, \mathbb{P})\text{-generic condition below } q \text{ exists.}\}$$

is projectively stationary.

Then $\mathbb{P}$ is ω_1-stationary preserving and σ-Baire.

2.2 A variant of the diamond principle in $[\omega_2]^\omega$

In the proof of Thm.1.5 we use a certain diamond principle in $[\omega_2]^\omega$. Here we prove that MM implies it. Recall that MM implies $2^{\omega_1} = \omega_2$. (See Foreman-Magidor-Shelah [1].) In fact we prove that $2^{\omega_1} = \omega_2$ implies it:

Lemma 2.6. *Assume that $2^{\omega_1} = \omega_2$. Let S be a stationary subset of E^2_0. Then there are $X \subseteq [\omega_2]^\omega$ and a sequence $\langle \mathcal{B}_x \mid x \in X\rangle$ with the following properties:*

(i) $\sup x \notin x$ *for each* $x \in X$, $\{\sup x \mid x \in X\} = S$, *and* $\sup \restriction X$ *is injective. ("*$\sup \restriction X$ *is injective" means that* $\sup x \neq \sup y$ *for all* $x, y \in X$ *with* $x \neq y$*.)*

(ii) $\mathcal{B}_x$ *is a countable family of subsets of* x *for each* $x \in X$.

(iii) For every sufficiently large regular cardinal θ, the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that

- $M \cap \omega_2 \in X$,
- $\mathcal{B}_{M\cap\omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$,

is projectively stationary.

Corollary 2.7. *Assume* MM. *Then for every stationary $S \subseteq E^2_0$ there are $X \subseteq [\omega_2]^\omega$ and a sequence $\langle \mathcal{B}_x \mid x \in X\rangle$ satisfying the properties (i)-(iii) in Lem.2.6.*

To prove Lem.2.6 we use the following fact, due to Shelah:

Fact 2.8 (Shelah). *If $2^{\omega_1} = \omega_2$, then $\diamondsuit_{\omega_2}(S)$ holds for every stationary $S \subseteq E^2_0$.*

First we prove the following lemma:

Lemma 2.9. *Assume that $2^{\omega_1} = \omega_2$. Let S be a stationary subset of E^2_0. Then there exist $X \subseteq [\omega_2]^\omega$ and a sequence $\langle b_x \mid x \in X\rangle$ such that*

(i') $\sup x \notin x$ *for each* $x \in X$, $\{\sup x \mid x \in X\} = S$, *and* $\sup \restriction X$ *is injective,*

(ii') $b_x \subseteq x$ *for each* $x \in X$

(iii') for every $B \subseteq \omega_2$ the set $\{x \in X \mid b_x = B \cap x\}$ is projectively stationary.

Proof. We may assume that $S \subseteq E^2_0 \setminus \omega_1$.

By Fact 2.8, $\diamondsuit_{\omega_2}(S)$ holds. Hence there exists a sequence $\langle H_\alpha, f_\alpha, b'_\alpha \mid \alpha \in S\rangle$ such that

(I) H_α is a stationary subset of ω_1, f_α is a function from $\alpha^{<\omega}$ to α, and $b'_\alpha \subseteq \alpha$.

(II) If H is a stationary subset of ω_1, F is a function from $\omega_2^{<\omega}$ to ω_2, and $B \subseteq \omega_2$, then there exists $\alpha \in S$ such that $H_\alpha = H$, $f_\alpha = F \restriction \alpha^{<\omega}$ and $b'_\alpha = B \cap \alpha$.

For each $\alpha \in S$, take $x_\alpha \in [\alpha]^\omega$ such that $\sup x = \alpha$, $x_\alpha \cap \omega_1 \in H_\alpha$ and x_α is closed under f_α. We can take such x_α because $\alpha \in E^2_0 \setminus \omega_1$ and H_α is stationary. Let $X := \{x_\alpha \mid \alpha \in S\}$. Moreover let $b_x := b'_{\sup x} \cap x$ for each $x \in X$. (Hence $b_{x_\alpha} = b'_\alpha \cap x_\alpha$ for each $\alpha \in S$.)

We show that these X and $\langle b_x \mid x \in X\rangle$ witness the lemma. Clearly they satisfy (i') and (ii'). We check (iii').

Fix $B \subseteq \omega_2$. It suffices to show that for every stationary $H \subseteq \omega_1$ and every function $F : \omega_2^{<\omega} \longrightarrow \omega$ there exists $x \in X$ such that $x \cap \omega_1 \in H$, x is closed under f and $b_x = B \cap x$.

Take an arbitrary stationary $H \subseteq \omega_1$ and an arbitrary function $F : \omega_2^{<\omega} \longrightarrow \omega_2$. Then there exists $\alpha \in S$ with $H_\alpha = H$, $f_\alpha = F \restriction \alpha^{<\omega}$ and $b'_\alpha = B \cap \alpha$. Then $x_\alpha \in X$. Moreover by the choice of x_α, $x_\alpha \cap \omega_1 \in H$, x_α is closed under F and $b_{x_\alpha} = b'_\alpha \cap x_\alpha = B \cap x_\alpha$. Hence x_α is what we seek.

This completes the proof. $\square$

Now we prove Lem.2.6:

Proof of Lem.2.6. For each $D \subseteq \mathrm{On} \times \mathrm{On}$ and $\gamma \in \mathrm{On}$, let $D[\gamma]$ denote the set $\{\beta \in \mathrm{On} \mid \langle \gamma, \beta \rangle \in D\}$. By Lem.2.9 there exist $X \subseteq [\omega_2]^\omega$ and a sequence $\langle d_x \mid x \in X \rangle$ such that

(i") $\sup x \notin x$ for each $x \in X$, $\{\sup x \mid x \in X\} = S$, and $\sup \restriction X$ is injective.

(ii") $d_x \subseteq x \times x$,

(iii") for every $D \subseteq \omega_2 \times \omega_2$ the set $\{x \in X \mid d_x = D \cap (x \times x)\}$ is projectively stationary.

For each $x \in X$ let $\mathcal{B}_x = \{d_x[\gamma] \mid \gamma \in x\}$.

We show that X and $\langle \mathcal{B}_x \mid x \in X \rangle$ witness Lem.2.6. Clearly (i) and (ii) hold. We check (iii).

Let θ be a sufficiently large regular cardinal. Take an arbitrary stationary $H \subseteq \omega_1$ and an arbitrary function $F : \mathcal{H}_\theta^{<\omega} \longrightarrow \mathcal{H}_\theta$. It suffices to find $M \in [\mathcal{H}_\theta]^\omega$ such that $M \cap \omega_1 \in H$, M is closed under F, $M \cap \omega_2 \in X$ and $\mathcal{B}_{M \cap \omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$

First take $N \subseteq \mathcal{H}_\theta$ such that $|N| = \omega_2 \subseteq N$, N is closed under F and $N \cap \mathcal{P}(\omega_2) \neq \emptyset$. Moreover take an enumeration $\langle B_\gamma \mid \gamma \in \omega_2 \rangle$ of $\mathcal{P}(\omega_2) \cap N$. For each $x \in X$ let

$$M_x := \mathrm{cl}_F(x \cup \{B_\gamma \mid \gamma \in x\}),$$

where $\mathrm{cl}_F(a)$ denotes the closure of a under F. Then let C be a set of all $x \in [\omega_2]^\omega$ such that $M_x \cap \omega_2 = x$ and $M_x \cap \mathcal{P}(\omega_2) = \{B_\gamma \mid \gamma \in x\}$. Finally let D be a subset of $\omega_2 \times \omega_2$ such that $D[\gamma] = B_\gamma$ for each $\gamma \in \omega_2$.

Note that C is a club in $[\omega_2]^\omega$. Hence, by (iii"), there exists $x \in X \cap C$ such that $x \cap \omega_1 \in H$ and $d_x = D \cap (x \times x)$. Then $M_x \in [\mathcal{H}_\theta]^\omega$, $M_x \cap \omega_1 = x \cap \omega_1 \in H$, and M_x is closed under F. Moreover

$$\begin{aligned} \mathcal{B}_{M_x \cap \omega_2} &= \mathcal{B}_x = \{d_x[\gamma] \mid \gamma \in x\} \\ &= \{D[\gamma] \cap x \mid \gamma \in x\} = \{B_\gamma \cap x \mid \gamma \in x\} \\ &= \{B \cap M_x \mid B \in \mathcal{P}(\omega_2) \cap M_x\}\,. \end{aligned}$$

Thus M_x is what we seek.

This completes the proof. $\square$

3 Proof of Thm.1.5

Before proving Thm.1.5 we present a poset to which we apply MM:

Definition 3.1. *For $S \subseteq E^2_0$ and a $\square(S)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in S \rangle$ let $\mathbb{P}(\vec{c})$ be the following poset:*

- *The base set of $\mathbb{P}(\vec{c})$ is S.*
- *$\alpha \leq_{\mathbb{P}(\vec{c})} \beta$ if and only if $\beta \in \mathrm{Lim}(c_\alpha) \cup \{\alpha\}$ for each $\alpha, \beta \in S$.*

For a filter g on $\mathbb{P}(\vec{c})$ let

$$c_g := \bigcup_{\alpha \in g} c_\alpha .$$

The following is easy:

Lemma 3.2. *Let S be a subset of E^2_0 and $\vec{c} = \langle c_\alpha \mid \alpha \in S \rangle$ be a $\square(S)$-sequence.*

(1) *If g is a filter on $\mathbb{P}(\vec{c})$, then c_g is a club in $\sup c_g$ of order type $\leq \omega_1$, $\mathrm{Lim}(c_g) \subseteq S$, and $c_\beta = c_g \cap \beta$ for each $\beta \in \mathrm{Lim}(c_g)$.*

(2) *Suppose that the following $(**)$ holds:*

   $(**)$ *$\mathbb{P}(\vec{c}) \setminus \gamma$ is dense in $\mathbb{P}(\vec{c})$ for every $\gamma < \omega_2$.*

   Let θ be a sufficiently large regular cardinal and M be an elementary submodel of $\langle \mathcal{H}_\theta, \in, \vec{c} \rangle$. Suppose also that g is an $(M, \mathbb{P}(\vec{c}))$-generic filter. Then $\sup c_g = \sup(M \cap \omega_2)$.

Now we prove Thm.1.5:

Proof of Thm.1.5. Assume MM. Our proof is composed of two steps. First we construct a $\square(E^2_0)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0 \rangle$ so that $\mathbb{P}(\vec{c})$ satisfies $(*)$ in Fact 2.5 and $(**)$ in Lem.3.2. Then, applying Fact 2.2 to $\mathbb{P}(\vec{c})$, we show that $\vec{c}$ can be extended to $\square(S)$-sequence for some $S \subseteq \mathrm{Lim}(\omega_2)$ with $S \cap E^2_1$ stationary.

(Step 1) Construction of $\vec{c}$.

First take a stationary partition $\langle T_\beta \mid \beta \in E^2_0 \rangle$ of E^2_0. For each $\beta \in E^2_0$ we can take $X_\beta \subseteq [\omega_2]^\omega$ and a sequence $\langle \mathcal{B}^\beta_x \mid x \in X_\beta \rangle$ with the following properties by Cor.2.7:

(i) $\sup x \notin X_\beta$ for each $x \in X_\beta$, $\{\sup x \mid x \in X_\beta\} = T_\beta$, and $\sup \restriction X_\beta$ is injective.

(ii) $\mathcal{B}^\beta_x$ is a countable family of subsets of x for each $x \in X_\beta$.

(iii) For every sufficiently large regular cardinal θ the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that

- $M \cap \omega_2 \in X_\beta$,
- $\mathcal{B}^\beta_{M\cap\omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$,

is projectively stationary.

By induction on $\alpha \in E^2_0$ we construct a $\square(E^2_0)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0\rangle$. First let $c_\omega = \omega$. Suppose that $\alpha \in E^2_0$ and that $\langle c_\beta \mid \beta \in E^2_0 \cap \alpha\rangle$ has been defined to be $\square(E^2_0 \cap \alpha)$-sequence. Then take c_α as follows:

Let $\beta_\alpha \in E^2_0$ be such that $\alpha \in T_{\beta_\alpha}$, and let x_α be the unique element of X_{β_α} with $\sup x_\alpha = \alpha$. If $\beta_\alpha \notin x_\alpha$ or there exists $\beta \in E^2_0 \cap x_\alpha$ with $\mathrm{Lim}(c_\beta) \not\subseteq x_\alpha$, then let c_α be an arbitrary unbounded subset of α of order type ω.

Suppose that $\beta_\alpha \in x_\alpha$ and that $\mathrm{Lim}(c_\beta) \subseteq x_\alpha$ for each $\beta \in E^2_0 \cap x_\alpha$. Then note that $\langle c_\beta \mid \beta \in E^2_0 \cap x_\alpha\rangle$ is a $\square(E^2_0 \cap x_\alpha)$-sequence. Let $\mathbb{P}_\alpha := \mathbb{P}(\langle c_\beta \mid \beta \in E^2_0 \cap x_\alpha\rangle)$. Note also that $\beta_\alpha \in \mathbb{P}_\alpha \subseteq x_\alpha$.

Recall that $\mathcal{B}^\beta_{x_\alpha}$ is a countable family of subsets of x_α. Hence we can take a filter g_α on $\mathbb{P}_\alpha$ such that

- $\beta_\alpha \in g_\alpha$,
- $g_\alpha \cap b \neq \emptyset$ for every $b \in \mathcal{B}^\beta_{x_\alpha}$ which is a maximal antichain in $\mathbb{P}_\alpha$.

If $\sup c_{g_\alpha} = \alpha$ then let $c_\alpha := c_{g_\alpha}$. Otherwise, take an unbounded $c \subseteq \alpha$ such that o.t.$(c) = \omega$ and $\beta_\alpha = \min c$, and let $c_\alpha := c_{\beta_\alpha} \cup c$.

This completes the choice of c_α. Using Lem.3.2 (1), it is easy to check that $\langle c_\beta \mid \beta \in E^2_0 \cap \alpha + 1\rangle$ is $\square(E^2_0 \cap \alpha + 1)$-sequence.

Now we have constructed a $\square(E^2_0)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0\rangle$. We show that $\mathbb{P}(\vec{c})$ satisfies $(*)$ and $(**)$:

Claim 1. *$\mathbb{P}(\vec{c})$ satisfies $(**)$ in Lem.3.2.*

Proof of Claim 1. Take an arbitrary $\beta^* \in E^2_0$ and an arbitrary $\gamma < \omega_2$. We must find $\alpha^* \in E^2_0 \setminus \gamma$ with $\alpha^* \leq_{\mathbb{P}(\vec{c})} \beta^*$.

Let θ be a sufficiently large regular cardinal. Because X_{β^*} is stationary in $[\omega_2]^\omega$, we can take $M \prec \langle \mathcal{H}_\theta, \in, \vec{c}\rangle$ such that $\beta^*, \gamma \in M$ and $M \cap \omega_2 \in X_{\beta^*}$. Let $\alpha^* := \sup(M \cap \omega_2)$. Clearly $\alpha^* \in E^2_0 \setminus \gamma$.

Note that $\beta_{\alpha^*} = \beta^*$ and $x_{\alpha^*} = M \cap \omega_2$. Hence $\beta_{\alpha^*} \in x_{\alpha^*}$ by the choice of M. Moreover $\mathrm{Lim}(c_\beta) \subseteq x_{\alpha^*}$ for every $\beta \in x_{\alpha^*}$ because $M \prec \langle \mathcal{H}_\theta, \in, \vec{c}\rangle$ and each c_β is a countable set. Then $\beta^* = \beta_{\alpha^*} \in \mathrm{Lim}(c_{\alpha^*})$ by the choice of c_{α^*}. Thus $\alpha^* \leq_{\mathbb{P}(\vec{c})} \beta^*$. ■(Claim 1)

Claim 2. *$\mathbb{P}(\vec{c})$ satisfies $(*)$ in Fact 2.5.*

Proof of Claim 2. Suppose that θ is a sufficiently large regular cardinal and that $\beta^* \in E^2_0 = \mathbb{P}(\vec{c})$. We prove that there are projectively stationary many $M \in [\mathcal{H}_\theta]^\omega$ for which a strongly $(M, \mathbb{P}(\vec{c}))$-generic condition below β^* exists.

Let X^* be the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that

- $\beta^*, \vec{c} \in M \prec \langle \mathcal{H}_\theta, \in\rangle$,

- $M \cap \omega_2 \in X_{\beta^*}$
- $\mathcal{B}^{\beta^*}_{M\cap\omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$.

Then X^* is projectively stationary by the choice of X_{β^*} and $\langle \mathcal{B}^{\beta^*}_x \mid x \in X_{\beta^*} \rangle$. It suffices to show that $\sup(M \cap \omega_2)$ is a strongly $(M, \mathbb{P}(\vec{c}))$-condition below β^* for each $M \in X^*$.

Fix $M \in X^*$ and let $\alpha^* := \sup(M \cap \omega_2)$. Note that $\beta_{\alpha^*} = \beta^*$ and $x_{\alpha^*} = M \cap \omega_2$. Hence $\beta_{\alpha^*} \in x_{\alpha^*}$, and $\mathrm{Lim}(c_\beta) \subseteq x_{\alpha^*}$ for each $\beta \in E^2_0 \cap x_{\alpha^*}$.

Here note that $\mathbb{P}_{\alpha^*} = \mathbb{P}(\vec{c}) \cap M$ and that g_{α^*} is an $(M, \mathbb{P}(\vec{c}))$-generic filter containing β^* by the choice of M and g_{α^*}. Then note also that $\sup c_{g_{\alpha^*}} = \sup(M \cap \omega_2) = \alpha^*$ by Lem.3.2 (2) and Claim 1. Hence $c_{\alpha^*} = c_{g_{\alpha^*}}$. Then α^* extends each element of g_{α^*}, which is an $(M, \mathbb{P}(\vec{c}))$-generic filter containing β^*. Therefore α^* is a strongly $(M, \mathbb{P}(\vec{c}))$-generic condition below β^*. ■(Claim 2)

Now we have constructed a $\square(E^2_0)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0 \rangle$ satisfying $(*)$ and $(**)$. ■(Step 1)

(Step 2) Extension of $\vec{c}$.

Let θ be a sufficiently large regular cardinal with $\theta^\omega = \theta$, and let Z be the set of all $N \in [\mathcal{H}_\theta]^{\omega_1}$ such that

(i) $N \prec \langle \mathcal{H}_\theta, \in, \vec{c} \rangle$,

(ii) N is internally approachable of length ω_1,

(iii) there exists an $(N, \mathbb{P}(\vec{c}))$-generic filter.

By Claim 2 and Fact 2.5, $\mathbb{P}(\vec{c})$ is ω_1-stationary preserving and σ-Baire. Hence Z is stationary in $[\mathcal{H}_\theta]^{\omega_1}$ by MM and Fact 2.2.

Note that $\sup(N \cap \omega_2) \in E^2_1$ for each $N \in Z$ because N is internally approachable of length ω_1. Hence $S' := \{\sup(N \cap \omega_2) \mid N \in Z\}$ is a stationary subset of E^2_1.

For each $\alpha \in S'$ choose $N_\alpha \in Z$ with $\sup(N_\alpha \cap \omega_2) \in S'$ and an $(N_\alpha, \mathbb{P}(\vec{c}))$-generic filter g_α. Moreover let $c_\alpha := c_{g_\alpha}$ for each $\alpha \in S'$. Note that $\sup c_\alpha = \alpha$ by Claim 1 and Lem.3.2 (2). Then, by Lem.3.2 (1), c_α is a club of α of order type ω_1, $\mathrm{Lim}(c_\alpha) \subseteq E^2_0$, and $c_\beta = c_\alpha \cap \beta$ for each $\beta \in \mathrm{Lim}(c_\alpha)$.

Now let $S := E^2_0 \cup S'$. Then $\langle c_\alpha \mid \alpha \in S \rangle$ is a $\square(S)$-sequence. ■(Step 2)

We have found $S \subseteq \mathrm{Lim}(\omega_2)$ such that $S \cap E^2_1$ is stationary and $\square(S)$ holds. This completes the proof of Thm.1.5. □

References

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